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On Two Correlation Inequalities for Potts Models

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The Fortuin–Kasteleyn random cluster representation of q-state Potts models is used to extend to every q two correlation inequalities proven previously only for even values of q.

KEY WORDS: Potts models; correlation inequalities; Fortuin–Kasteleyn random cluster model.

In ref. 1 some correlation inequalities for q-state ferromagnetic Potts models are proven and used to derive results about surface tensions. For technical reasons these inequalities are proven there only when q is an even number. Here I provide alternative proofs of these correlation inequalities which apply to any value of q. As a consequence, the results in ref. 1 for the surface tensions (Theorems 4 and 5 in that paper) are valid with this generality.

Before stating and proving the results, I introduce the basic notation. Let Λ be a finite, completely ordered set (the points in Λ are called sites) and to each $i \in \Lambda$ attach a spin σ_i, which may assume the q values $\{1,..., q\}$, where $q \geqslant 2$. The energy of the configuration $\sigma \in \{1,..., q\}^\Lambda$ is given by

$$H(\sigma) = -\sum_{i<j} J_{ij} \delta(\sigma_i, \sigma_j)$$

where $\delta(a, b)$ is equal to 1 if $a = b$ and 0 otherwise. $J_{ij} \geqslant 0$ are the ferromagnetic interactions. $\langle \cdot \rangle$ will represent the expected value with

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respect to the corresponding Gibbs measure (the temperature is arbitrary and may be thought of as absorbed in the J_{ij}):

$$\langle f(\sigma)\rangle = \frac{\sum_\sigma f(\sigma)\exp[-H(\sigma)]}{\sum_\sigma \exp[-H(\sigma)]}$$

for any observable f: $\{1,..., q\}^\Lambda \to \mathbb{R}$.

Theorem. Under the conditions above, if A and B are two subsets of Λ, a, $b \in \{1,..., q\}$ and $a \neq b$, then (i)

$$\left\langle \prod_{i\in A} \delta(\sigma_i, a) \prod_{j\in B} \delta(\sigma_j, a)\right\rangle \geqslant \left\langle \prod_{i\in A} \delta(\sigma_i, a)\right\rangle \left\langle \prod_{j\in B} \delta(\sigma_j, a)\right\rangle$$

and (ii)

$$\left\langle \prod_{i\in A} \delta(\sigma_i, a) \prod_{j\in B} \delta(\sigma_j, b)\right\rangle \leqslant \left\langle \prod_{i\in A} \delta(\sigma_i, a)\right\rangle \left\langle \prod_{j\in B} \delta(\sigma_j, b)\right\rangle$$

The proof of these results will be based on the use of the Fortuin–Kasteleyn representation for Potts models.[4,5] This construction was transformed into a powerful tool for obtaining rigorous results in ref. 2, which should be consulted for a systematic exposition of the subject. (See also ref. 3.) Next I review the definitions and relations that will be needed (for proofs see ref. 2).

Given $\Gamma \subset \Lambda$, define $\mathscr{L}(\Gamma) = \{\{i, j\}: i, j \in \Gamma\}$, $\mathscr{L} = \mathscr{L}(\Lambda)$. Each pair $\{i, j\}$, $i, j \in \Lambda$ is called a bond, and $\mathscr{L}(\Gamma)$ is the set of bonds linking sites in Γ. The Fortuin–Kasteleyn random cluster model is introduced by randomly choosing a set of bonds that will be said to be occupied (the others being said to be vacant). Let $\mathscr{S}$ be the random set of occupied bonds. The random cluster probability measure W is defined by taking the probability $W(\mathscr{S} = S)$ proportional to

$$\prod_{\{i,j\}\in S} p_{ij} \cdot \prod_{\{i,j\}\in \mathscr{L}\setminus S} (1-p_{ij}) \cdot q^{c(S)}$$

where $p_{ij} = 1 - \exp(-J_{ij})$ and $c(S)$ is the number of clusters into which Λ is partitioned by S if we declare two sites i and j to belong to the same cluster if there exists a chain of bonds (i, k_1), (k_1, k_2),..., (k_{n-1}, k_n), (k_n, j) all of which belong to S (i.e., are occupied). In the terminology of ref. 2, W corresponds to the random cluster model with free boundary conditions. It is related to the Gibbs measure for the Potts model by relation (2.7) in ref. 2: for any observable f,

$$\langle f(\sigma)\rangle = \sum_{S\subset\mathscr{L}} W(\mathscr{S} = S)\, E_S(f(\sigma)) \tag{1}$$

Here, for each S, $E_S(\cdot)$ is the average over the spins σ_i obtained by constraining all the spins in each of the clusters of S to assume the same value and choosing these values for the different clusters independently according to the uniform distribution on $\{1,..., q\}$.

A basic ingredient will be the FKG property of W, proved in ref. 2. Consider the partial order on the set $\mathscr{C}$ of subsets of $\mathscr{L}$ defined by inclusion ($S_1 \leqslant S_2$ if and only if $S_1 \subset S_2$). Then for any pair $F, G: \mathscr{C} \to \mathbb{R}$ of nondecreasing functions,

$$\left[\sum_S F(S)\, W(\mathscr{S}=S)\right]\left[\sum_S G(S)\, W(\mathscr{S}=S)\right] \leqslant \sum_S F(S)\, G(S)\, W(\mathscr{S}=S) \tag{2}$$

In order to prove part (i) of the theorem, we first observe that it is enough to consider the case in which A and B are disjoint, since other cases reduce easily to this. By the properties of Gibbs measures, part (i) is equivalent to

$$\left\langle \prod_{j \in B} \delta(\sigma_j, a) \right\rangle_A \geqslant \left\langle \prod_{j \in B} \delta(\sigma_j, a) \right\rangle$$

Here $\langle \cdot \rangle_A$ is the expectation corresponding to the Gibbs measure on $\{1,..., q\}^{\Lambda \setminus A}$, with boundary condition of type a in A, i.e., with energy given by

$$H_A(\sigma) = -\sum_{\substack{i<j\\ i,j \in \Lambda \setminus A}} J_{ij}\delta(\sigma_i, \sigma_j) - \sum_{\substack{i \in \Lambda \setminus A\\ j \in A}} J_{ij}\delta(\sigma_i, a)$$

As remarked in ref. 2, $\langle \cdot \rangle_A$ also has a representation in terms of the random cluster model:

$$\langle f(\sigma) \rangle_A = \sum_S W_A(\mathscr{S}=S)\, E_S^{A,a}(f(\sigma)) \tag{3}$$

In this expression W_A is obtained from W by conditioning all the bonds that link points in A to be occupied. $E_S^{A,a}(\cdot)$ has a definition similar to that of $E_S(\cdot)$, except that the spins on clusters that intersect A assume the value a with probability 1. It follows from the FKG relation (2), by taking G as the indicator function of the event $\{\mathscr{L}(A) \subset \mathscr{S}\}$, that for any nondecreasing F,

$$\sum_S F(S)\, W(\mathscr{S}=S) \leqslant \sum_S F(S)\, W_A(\mathscr{S}=S) \tag{4}$$

Part (i) now follows easily from (1), (3), and (4). For this purpose let $N(S)$ be the number of clusters into which B is divided when the set of occupied bounds is S. Let $N_A(S)$ be the number of those clusters above that do not intersect A. Then

$$\begin{aligned}\left\langle \prod_{j\in B} \delta(\sigma_j, a)\right\rangle_A &= \sum_S W_A(\mathcal{S}=S)(1/q)^{N_A(S)}\\ &\geqslant \sum_S W_A(\mathcal{S}=S)(1/q)^{N(S)}\\ &\geqslant \sum_S W(\mathcal{S}=S)(1/q)^{N(S)}\\ &= \left\langle \prod_{j\in B} \delta(\sigma_j, a)\right\rangle\end{aligned}$$

The use of (4) in the second inequality is justified since $(1/q)^{N(S)}$ is increasing in S.

We turn now to the proof of part (ii). Clearly we can consider A and B disjoint, since otherwise there is nothing to prove. Let us denote by $\{A \leftrightarrow B\}$ the event that some cluster intersects A and B. Now, by (1),

$$\begin{aligned}&\left\langle \prod_{i\in A} \delta(\sigma_i, a) \prod_{j\in B} \delta(\sigma_j, b)\right\rangle\\ &\quad = \sum_{S\subset\mathscr{L}} W(\mathcal{S}=S) E_S\left[\prod_{i\in A} \delta(\sigma_i, a) \prod_{j\in B} \delta(\sigma_j, b)\right]\\ &\quad = \sum_{S\in\{A\leftrightarrow B\}^c} W(\mathcal{S}=S) E_S\left[\prod_{i\in A} \delta(\sigma_i, a)\right] E_S\left[\prod_{j\in B} \delta(\sigma_j, b)\right] \qquad (5)\end{aligned}$$

The proof will be finished once we show that the rhs of (5) is not greater than

$$\begin{aligned}&\left\{\sum_{S\subset\mathscr{L}} W(\mathcal{S}=S) E_S\left[\prod_{i\in A} \delta(\sigma_i, a)\right]\right\}\left\{\sum_{S\subset\mathscr{L}} W(\mathcal{S}=S) E_S\left[\prod_{j\in B} \delta(\sigma_j, b)\right]\right\}\\ &\quad = \left\langle \prod_{i\in A} \delta(\sigma_i, a)\right\rangle\left\langle \prod_{j\in B} \delta(\sigma_i, b)\right\rangle\end{aligned}$$

The idea behind the proof of this inequality is the fact that once the clusters of A do not touch B, the spins inside B only "see" the boundary of these clusters, which are vacant bonds. Then, by FKG the number of clusters in B increases and it becomes more difficult for all the spins in B to take the same value b.

To make these statements precise, we need some definitions. Given $S \subset \mathscr{L}$, set $\Lambda(S) = \{i \in \Lambda: \{i, j\} \in S \text{ for some } j\}$. Let $\bar{S}$ denote S plus its boundary, i.e., $\bar{S} = \{\{i, j\}: i \in \Lambda(S), j \in \Lambda\}$. Let $\mathscr{I}$ be the random set of sites that belong to clusters which intersect A. $\mathscr{T}$ will denote the random set of occupied bonds that form these clusters: $\mathscr{T} = \mathscr{S} \cap \mathscr{L}(\mathscr{I})$. Now the rhs of (5) is equal to

$$\sum_{\substack{I:\, A \subset I \\ B \cap I = \varnothing}} \sum_{T \subset \mathscr{L}(I)} \sum_{V \subset \mathscr{L}(\Lambda \setminus I)} W(\mathscr{I} = I, \mathscr{T} = T, \mathscr{S} \setminus T = V)$$
$$\times E_T\left[\prod_{i \in A} \delta(\sigma_i, a)\right] E_V\left[\prod_{j \in B} \delta(\sigma_j, b)\right] \tag{6}$$

But for any I, T, and V

$$\begin{aligned} &W(\mathscr{I} = I, \mathscr{T} = T, \mathscr{S} \setminus T = V) \\ &\quad = W(\mathscr{I} = I, \mathscr{T} = T)\, W(\mathscr{S} \cap \mathscr{L}(\Lambda \setminus I) = V \mid \mathscr{I} = I, \mathscr{T} = T) \\ &\quad = W(\mathscr{I} = I, \mathscr{T} = T)\, W(\mathscr{S} \cap \mathscr{L}(\Lambda \setminus I) = V \mid \mathscr{S} \cap \overline{\mathscr{L}(I)} = \varnothing) \\ &\quad = W(\mathscr{I} = I, \mathscr{T} = T)\, W(\mathscr{S} = V \mid \mathscr{S} \cap \overline{\mathscr{L}(I)} = \varnothing) \end{aligned} \tag{7}$$

The second equality follows from direct computations, which show that the second factor on either side of this equality is equal to

$$\left(\prod_{\{i,j\} \in V} p_{ij}\right)\left[\prod_{\{i,j\} \in \mathscr{L}(\Lambda \setminus I) \setminus V} (1 - p_{ij})\right] q^{c_{\Lambda \setminus I}(V)}$$

where $c_{\Lambda \setminus I}(V)$ is the number of clusters into which $\Lambda \setminus I$ is partitioned by V. We apply now the FKG relation (2) with F as minus the indicator function of $\{\mathscr{S} \cap \overline{\mathscr{L}(I)} = \varnothing\}$ and

$$G(S) = E_S\left[\prod_{j \in B} \delta(\sigma_j, b)\right] = (1/q)^{N(S)}$$

It follows that for fixed I

$$\begin{aligned} &\sum_{V \subset \mathscr{L}(\Lambda \setminus I)} W(\mathscr{S} = V \mid \mathscr{S} \cap \overline{\mathscr{L}(I)} = \varnothing)\, E_V\left[\prod_{j \in B} \delta(\sigma_j, b)\right] \\ &\quad \leqslant \sum_{V \subset \mathscr{L}} W(\mathscr{S} = V)\, E_V\left[\prod_{j \in B} \delta(\sigma_j, b)\right] \\ &\quad = \left\langle \prod_{j \in B} (\sigma_j, b) \right\rangle \end{aligned} \tag{8}$$

Using (7) and (8), we see that the expression (6) is not greater than

$$\sum_{\substack{I:\, A \subset I \\ B \cap I = \varnothing}} \sum_{T \subset \mathscr{L}(I)} \left\{ W(\mathscr{I} = I, \mathscr{T} = T)\, E_T\left[\prod_{i \in A} \delta(\sigma_i, a)\right] \right.$$

$$\left. \times \sum_{V \subset \mathscr{L}(A \setminus I)} W(\mathscr{S} = V \mid \mathscr{S} \cap \overline{\mathscr{L}(I)} = \varnothing)\, E_V\left[\prod_{j \in B} \delta(\sigma_j, b)\right] \right\}$$

$$\leqslant \left\langle \prod_{j \in B} \delta(\sigma_j, b) \right\rangle \sum_{\substack{I:\, A \subset I \\ B \cap I = \varnothing}} \sum_{T \subset \mathscr{L}(I)} W(\mathscr{I} = I, \mathscr{T} = T)\, E_T\left[\prod_{i \in A} \delta(\sigma_i, a)\right]$$

$$\leqslant \left\langle \prod_{j \in B} \delta(\sigma_j, b) \right\rangle \left\langle \prod_{i \in A} \delta(\sigma_i, a) \right\rangle$$

This finishes the proof of the theorem. ▮

In ref. 1 inequalities (i) and (ii) are proven when q is even and the interactions are between nearest neighbors on a cubic lattice Z^d ($J_{ij} = 0$ if i and j are not nearest neighbors). There is nevertheless one sense in which the results in ref. 1 are more general than the ones here: they include some positive external fields H_i and K_i in the interaction. In our case this would amount to taking

$$H(\sigma) = -\sum_{i<j} J_{ij} \delta(\sigma_i, \sigma_j) - \sum_i H_i \delta(\sigma_i, a) - \sum_i K_i\, \delta(\sigma_i, b)$$

With the techniques used here we can handle the case in which only the first extra term is present, i.e., $H_i \geqslant 0$, $K_i = 0$. To prove part (i) of the theorem, it is enough to consider a ghost site g and $J_{ig} = H_i$ (take $g > i$ for all $i \in A$, so that the J_{gi} are irrelevant). The external field is then transformed into the boundary condition $\sigma_g = a$. The proof of (i) is then essentially the same, but (1) has to be modified to account for the effect of the boundary condition. To prove (ii), consider also the ghost site as above. The proof is again basically the same as used before, but the roles of A and B have to be interchanged (i.e., $\mathscr{I}$ will be the set of sites connected to B, etc.). It is not clear whether one can use the techniques in this paper to deal with the cases of two or more types of external fields.

The extensions of Theorems 4 and 5 of ref. 1 to all values of q do not depend on the above remarks, since their proofs rely completely on the correlation inequality (ii) for Potts models with no external fields and with free boundary conditions.

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